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Rare pionium decays and pion polarizability

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Abstract. We calculate the decay of pionium atoms into two photons. The pion polarizabilities give rise to a 10% correction to the corresponding decay width for pointlike pions. This opens the possibility to obtain the difference between the electric and magnetic polarizability of the charged pion from a future measurement of the branching fraction of pionium into two photons. For such an experiment the $\pi\pi$-scattering lengths would have to be known to better than 5% precision. We also comment on the contribution of the axial anomaly to the decay of pionium into $\gamma\pi^0$.

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The response of a composite particle to an external electromagnetic field is described by its electric (α) and magnetic (β) polarizabilities [1]. They are fundamental quantities whose understanding is of great importance in any model or theory of the strong interaction. In particular, the polarizabilities of the pion allow for a clean test of Chiral Perturbation Theory (ChPT). To one loop order only two of the ten low energy constants at $\mathcal{O}(p^4)$ contribute and ChPT predicts $\alpha_{\pi^\pm} = -\beta_{\pi^\pm} = 2.8 \cdot 10^{-4}$ fm^3 [1].[1] To this order one has $\alpha_\pi = -\beta_\pi$ because the lowest order coupling to the electromagnetic field is proportional to $F_{\mu\nu}F^{\mu\nu} \sim (\boldsymbol{E}^2 - \boldsymbol{B}^2)$. Since higher order corrections are suppressed this relation is expected to hold approximately to all orders. More recently, Bürgi performed a two loop calculation and obtained $\alpha_{\pi^\pm} = 2.4 \pm 0.5$ and $\beta_{\pi^\pm} = -2.1 \pm 0.5$ [2].

Experimentally both of these quantites are measured using Compton scattering off pions. However, since a pion target is not available, the process is only indirectly accessible when embedded in other reactions. As a consequence, experiments are difficult and the polarizabilities are relatively poorly known. For example one of the experiments done is to use radiative pion scattering off a heavy nucleus, $\pi^- Z \to \pi^- Z\gamma$. In this case the pion scatters off a virtual photon in the Coulomb field of the nucleus (Primakoff scattering). This method was used in the Serpukhov experiment [3], resulting in $\alpha_{\pi^\pm} = -\beta_{\pi^\pm} = 6.8 \pm 1.4 \pm 1.2$. A similar way is to use the radiative pion photoproduction process, $\gamma p \to \gamma\pi^+ n$, where the incoming photon scatters off a virtual pion. Such an experiment was performed at the Lebedev Institute [4] with the result $\alpha_{\pi^\pm} = -\beta_{\pi^\pm} = 20 \pm 12$. Both experiments are in variance with the results from ChPT [1,2]. Furthermore, they suffer from a limited statistical accuracy and systematic errors. The extrapolation from the actual data to the Compton scattering amplitude in the case of radiative pion photoproduction has recently been studied in [5] and a new experiment has been proposed at Mainz. An overview of proposed Primakoff experiments is given in [6].

In a second type of experiment, one makes use of the crossed channel reaction $\gamma\gamma \to \pi\pi$ [7,8] and attempts to extract the polarizabilities from the cross-section. This reaction also has the advantage of being able to determine the neutral pion polarizability as well. However, it suffers from the difficulties of having to deal with strong final state interaction. The most recent analysis [8] is consistent with the result from ChPT but the experimental error bars can accomodate differences in polarizability calculations by a factor of two.

In light of the present discrepancy between theory and experiment, we propose to study the decay mode of pionium into two photons. The formation of pionium atoms has been observed recently [9]. They decay in more than 99% of the time into two neutral pions. Furthermore, pionium can decay into two photons and $\gamma\pi^0$ as well. The DIRAC Collaboration at CERN aims to measure the pionium lifetime with 10% precision in order to extract the $\pi\pi$-scattering lengths [10]. Recently, the decay of pionium atoms has been studied using nonrelativistic effective field theories [11] and a general expression that allows the extraction of the $\pi\pi$-scattering lengths has been given in this framework as well [12]. In this note we point out that if the next generation of pionium experiments can measure

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[1] In the following, we will express the polarizabilities in units of 10^{-4} fm^3.

the branching ratio $\Gamma(\text{pionium} \to \gamma\gamma)/\Gamma_{tot}$ sufficiently accurate, it should be able to provide a clean determination of the charged pion polarizability. Below, we will calculate the decay pionium$\to \gamma\gamma$. The theoretical assumptions required are crossing symmetry and the usual formulation of bound states in quantum field theory. We show that the pion polarizability provides corrections of the order of 10% to the point particle result for $\Gamma(\text{pionium} \to \gamma\gamma)$.

We first consider Compton scattering off charged pions,

$$\gamma(q_1, \epsilon_1) + \pi^+(p_1) \to \gamma(q_2, \epsilon_2) + \pi^+(p_2) \ . \qquad (1)$$

The polarizabilities are defined by the threshold expansion of the Compton scattering amplitude in the laboratory frame,

$$\mathcal{M}_{\gamma\pi\to\gamma\pi} = 8\pi m_\pi \left[\left(-\frac{\alpha}{m_\pi} + \omega_1\omega_2\alpha_\pi\right)\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_2} + (\boldsymbol{\epsilon_1}\times\boldsymbol{q_1})\cdot(\boldsymbol{\epsilon_2}\times\boldsymbol{q_2})\beta_\pi\right], \qquad (2)$$

where $\alpha = e^2/4\pi$ is the electromagnetic fine structure constant and m_π is the charged pion mass. $\omega_{1(2)}$, $\boldsymbol{q}_{1(2)}$, and $\boldsymbol{\epsilon}_{1(2)}$ are the energy, momentum, and polarization vectors of the photon in the initial (final) state, respectively. The Compton tensor $M_{\mu\nu}$ is defined by

$$\mathcal{M} = ie^2 M_{\mu\nu}\epsilon_1^\mu \epsilon_2^\nu \ . \qquad (3)$$

The structure of $M_{\mu\nu}$ is given by [13–15],

$$\begin{aligned} M_{\mu\nu} &= \int d^4x\, e^{iq_2\cdot x}\langle\pi^+(p_2)|T[J_\nu^{EM}(x)J_\mu^{EM}(0)]|\pi^+(p_1)\rangle \\ &= -\frac{T_\mu(p_1,p_1+q_1)T_\nu(p_2+q_2,p_2)}{(p_1+q_1)^2-m_\pi^2} \\ &\quad -\frac{T_\mu(p_2-q_1,p_2)T_\nu(p_1,p_1-q_2)}{(p_1-q_2)^2-m_\pi^2} \\ &\quad +2g_{\mu\nu} + \frac{1}{3}\langle r_\pi^2\rangle(q_1^2 g_{\mu\nu} - q_{1\mu}q_{1\nu} + q_2^2 g_{\mu\nu} - q_{2\mu}q_{2\nu}) \\ &\quad -\frac{m_\pi}{\alpha}(\alpha_\pi-\beta_\pi)(q_1\cdot q_2 g_{\mu\nu} - q_{2\mu}q_{1\nu}) + \ldots, \end{aligned} \qquad (4)$$

where

$$T_\mu(p_i,p_f) = (p_i+p_f)_\mu\left[1+\frac{1}{6}\langle r_\pi^2\rangle q^2\right] + q_\mu\frac{1}{6}\langle r_\pi^2\rangle(p_i^2-p_f^2) + \ldots \ , \qquad (5)$$

is the electromagnetic vertex of the pion and $q = p_f - p_i$. The dots stand for higher order loop corrections which we will neglect in the following. The polarizabilities appear only in the combination $\alpha_\pi - \beta_\pi$ since to this order the coupling to the electromagnetic field is proportional to $F_{\mu\nu}F^{\mu\nu} \sim (\boldsymbol{E}^2 - \boldsymbol{B}^2)$.

For the pionium decay into two photons, we need the crossed amplitude $\tilde{M}_{\mu\nu}$ for the reaction

$$\pi^+(p_1) + \pi^-(p_2) \to \gamma(q_1,\epsilon_1) + \gamma(q_2,\epsilon_2) \ , \qquad (6)$$

which is readily obtained from (4) by the substitutions $(p_2 \to -p_2)$ and $(q_1 \to -q_1)$,

$$\begin{aligned} \tilde{M}_{\mu\nu} &= -\frac{T_\mu(p_1,p_1-q_1)T_\nu(q_2-p_2,-p_2)}{(p_1-q_1)^2-m_\pi^2} \\ &\quad -\frac{T_\mu(q_1-p_2,-p_2)T_\nu(p_1,p_1-q_2)}{(p_1-q_2)^2-m_\pi^2} \\ &\quad +2g_{\mu\nu} + \frac{1}{3}\langle r_\pi^2\rangle(q_1^2 g_{\mu\nu} - q_{1\mu}q_{1\nu} + q_2^2 g_{\mu\nu} - q_{2\mu}q_{2\nu}) \\ &\quad +\frac{m_\pi}{\alpha}(\alpha_\pi-\beta_\pi)(q_1\cdot q_2 g_{\mu\nu} - q_{2\mu}q_{1\nu}) + \ldots \ . \end{aligned} \qquad (7)$$

Using (7), we calculate the width for the decay pionium$\to \gamma\gamma$ at leading order. The decay width Γ is given by

$$d\Gamma = |\psi(0)|^2 \frac{1}{4m_\pi^2}\frac{1}{2}|\mathcal{M}|^2 \frac{d^3q_1}{2q_{10}(2\pi)^3}\frac{d^3q_2}{2q_{20}(2\pi)^3}(2\pi)^4 \times\delta^4(p_1+p_2-q_1-q_2) \ , \qquad (8)$$

where $\psi(0)$ is the Coulomb wave function for the $\pi^+\pi^-$ bound state at the origin. Evaluating (8) in the pionium rest frame, the pion charge radius does not contribute and we obtain

$$\Gamma = \frac{2\pi\alpha^2}{m_\pi^2}|\psi(0)|^2\left[1+\frac{m_\pi^3}{\alpha}(\alpha_\pi-\beta_\pi)\right]^2 \ . \qquad (9)$$

From the Coulomb wave function, we have

$$|\psi(0)|^2 = \frac{1}{\pi a_B^3} = \frac{1}{\pi}\left(\frac{m_\pi\alpha}{2}\right) \ , \qquad (10)$$

and finally obtain

$$\Gamma(\text{pionium}\to\gamma\gamma) = \frac{m_\pi\alpha^5}{4}\left[1+\frac{m_\pi^3}{\alpha}(\alpha_\pi-\beta_\pi)\right]^2 \ . \qquad (11)$$

The corresponding formula without the polarizability corrections has already been given in [16]. Using the central value of $\alpha_{\pi^\pm} = -\beta_{\pi^\pm} = 6.8 \pm 1.4 \pm 1.2$ [3], we find

$$\Gamma(\text{pionium}\to\gamma\gamma) = 0.723\,\text{meV}\cdot[1+0.132+0.004] \qquad (12)$$

The correction of the polarizability to the pointlike result is of the order 10% which allows for a clean measurement of the pion polarizability.[2] By far the dominating decay of the pionium, however, is the one into two neutral pions. The corresponding decay width is given by [17] (see also [18]),

[2] A similar sensitivity to the polarizabilities is observed in the reaction $\gamma\gamma \to \pi^+\pi^-$ (see e.g. [2]). The pionium decay, however, has the advantage that the pions annihilate at rest and we obtain the amplitude directly at threshold.

$$\Gamma(\text{pionium} \to \pi^0\pi^0) = |\psi(0)|^2 \frac{16\pi}{9}(a_0^0 - a_0^2)^2 \sqrt{\frac{\Delta m_\pi}{m_\pi}} \,, \tag{13}$$

where a_0^0 (a_0^2) is the s-wave $\pi\pi$-scattering length for isospin 0 (2) and $\Delta m_\pi = m_\pi - m_{\pi^0}$. Approximating $\Gamma_{tot} \approx \Gamma(\text{pionium} \to \pi^0\pi^0)$, we obtain the branching ratio

$$\frac{\Gamma(\text{pionium}\to\gamma\gamma)}{\Gamma_{tot}} = \left\{\frac{9\alpha^2}{8}[m_\pi(a_0^0 - a_0^2)]^{-2}\left(\frac{\Delta m_\pi}{m_\pi}\right)^{-1/2}\right\} \times \left[1 + \frac{m_\pi^3}{\alpha}(\alpha_\pi - \beta_\pi)\right]^2 \,, \tag{14}$$

where the factor in the curly bracket is about 0.28%. As seen in (14), the pionium wave function has dropped out and the associated uncertainties are eliminated. The main source of uncertainty is now the value of the $\pi\pi$-scattering lengths. The current experimental value for $(a_0^0 - a_0^2) = (0.288 \pm 0.051)m_\pi^{-1}$ is taken from [19]. In order to be able to extract the pion polarizabilities, the uncertainty introduced by the $\pi\pi$-scattering lengths has to be smaller than the 10% correction the polarizabilities give to the pointlike result. Consequently, one needs to know $(a_0^0 - a_0^2)$ with a 5% precision or better which appears to be an attainable goal for the future. When the smaller value $\alpha_{\pi^\pm} - \beta_{\pi^\pm} = 4.5 \pm 1.0$ [2] predicted by ChPT is used, $(a_0^0 - a_0^2)$ has to be known to higher precision.

We also note that for the actual extraction of $\alpha_{\pi^\pm} - \beta_{\pi^\pm}$ from experiment, the systematic inclusion of radiative corrections in (14) is needed. If, e.g., the one-loop amplitude for $\pi^+\pi^- \to \gamma\gamma$ (see [2]) instead of the tree-level result is used, we obtain a correction of 0.042 to the 1 in the square bracket of (9, 11, 12, 14). This correction does not affect the sensitivity to $\alpha_{\pi^\pm} - \beta_{\pi^\pm}$. However, it is not negligible compared to the effect of the polarizabilities and therefore needed for a sensible extraction from experiment. Furthermore, there are corrections from the strong interaction to the pionium wave function which cancels in (14) to leading order (See e.g. [12] and references therein). However, since the suggested measurement is beyond the present experimental capabilities, such a higher order calculation would be premature.

We next turn to the decay of pionium into $\gamma\pi^0$. This branch is to leading order determined by the axial anomaly of QCD. In principle this opens the possibility of measuring the anomaly via the decay pionium$\to \gamma\pi^0$ which, however, is outside the scope of this note. The amplitude for $\pi^+(p_1) + \pi^-(p_2) \to \gamma(q_1, \epsilon_1) + \pi^0(q_2)$ is given by [20,21]

$$\mathcal{M} = \frac{e}{4\pi^2 F_\pi^3}\epsilon^{\mu\nu\alpha\beta}\epsilon_{1\mu} q_{2\nu} p_{1\alpha} p_{2\beta} \,. \tag{15}$$

Naively one would expect this to be order α of the dominant two neutral pion decays. A close examination of (15) reveals that this mode is suppressed by the relative three momentum of the initial state pions. Since at threshold where pionium is formed $p_1 = p_2 = (m_\pi, \mathbf{0})$, this contribution vanishes and the anomaly contributes only at $\mathcal{O}(\boldsymbol{p})$. Moreover, this result justifies our approximation of the total width by $\Gamma(\text{pionium} \to \pi^0\pi^0)$ in (14).

In conclusion, we have presented the calculation of the contribution of pion polarizablities to the branching ratio of the rare two photons decays of pionium. As seen in (14), the only uncertainty involves the values of the $\pi\pi$-scattering lengths which one can determine accurately. In fact, these scattering lengths will be measured by a first generation pionium experiment at CERN [10]. If the scattering lengths can be determined to better than 5% accuracy, the branching ratio into two photons offers a clean way of determining the charged pion polarizabilites. The remaining theoretical uncertainties involve higher order loop effects and are expected to be small. In addition, this measures $\alpha_{\pi^\pm} - \beta_{\pi^\pm}$ at threshold which is accessible currently only by extrapolation and hence plagued with uncertainties. We are aware that rare decays of pionium are dauntingly difficult experiments. However, in view of the importance of polarizabilities in hadronic physics and the clean nature of the decay we believe it is worthwhile pursuing it vigorously.

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